

A note on the optimality of decomposable entanglement witnesses and completely entangled subspaces

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Abstract. Entanglement witnesses (EWs) constitute one of the most important entanglement detectors in quantum systems. Nevertheless, their complete characterization, in particular with respect to the notion of optimality, is still missing, even in the decomposable case. Here we show that for any qubit-qunit decomposable EW (DEW) W the three statements are equivalent: (i) the set of product vectors obeying $\langle e, f | W | e, f \rangle = 0$ spans the corresponding Hilbert space, (ii) W is optimal, (iii) $W = Q^\Gamma$ with Q denoting a positive operator supported on a completely entangled subspace (CES) and Γ standing for the partial transposition. While, implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are known, here we prove that (iii) implies (i). This is a consequence of a more general fact saying that product vectors orthogonal to any CES in $\mathbb{C}^2 \otimes \mathbb{C}^n$ span after partial conjugation the whole space. On the other hand, already in the case of $\mathbb{C}^3 \otimes \mathbb{C}^3$ Hilbert space, there exist DEWs for which (iii) does not imply (i). Consequently, either (i) does not imply (ii), or (ii) does not imply (iii), and the above transparent characterization obeyed by qubit-qunit DEWs, does not hold in general.

1. Introduction

Entanglement witnesses (EW) [1, 2] provide one of the best known methods of entanglement detection in composite (bipartite and multipartite) quantum systems (see the recent review [3] for other methods). These are Hermitian operators which, on one hand, have nonnegative mean values in all separable states and, on the other hand, they must have negative mean values in some entangled states.

The particular importance of EWs in detection of entanglement stems from several facts. First of all, we know that they give rise to a necessary and sufficient condition for separability [1] (see also [4] for the multipartite case). Precisely, given ϱ is separable if and only if $\langle W \rangle_{\varrho} \equiv \text{Tr}(W\varrho) \geq 0$ for all EWs or, equivalently, ϱ is entangled if and only if $\langle W \rangle_{\varrho} < 0$ for at least one such W . For the above reasons it is not feasible to check the "if" part of this criterion, nevertheless it still gives a strong necessary condition for separability. Then, as it was first stressed in Ref. [2], since EWs are Hermitian operators, it is clear that they correspond to some quantum observables and therefore the above criterion is applicable in experiment (see e.g. [5, 6]). Finally, there is a whole bunch of works indicating their quantitative meaning (see e.g. [7, 8, 9, 10, 11]). More precisely, mean values of entanglement witnesses not only serve as entanglement detectors, but also can tell us how much entangled the state is.

Although the above extensive literature as well as Refs. [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]) is aiming at studying their properties and providing methods of construction it seems that still much can and should to be said about EWs. In particular, complete characterization and classification of EWs is far from satisfactory. The structure of the so-called optimal EWs (in the sense of Ref. [12], see also below for the definition), even in the decomposable case, is still unknown. The importance of this problem stems from the fact that the above separability criterion can be restated using optimal EWs only. This is because every EW which is not optimal can be optimized [12]. Therefore it is of great importance to characterize the set of EWs with respect to their optimality. The early attempts to achieve this goal were discussed already in [12].

The main purpose of this note is to go towards solving the above problems. We investigate few notions connected to the optimality of the decomposable entanglement witnesses. In particular, we show that in the case of qubit-qunit Hilbert spaces, a more exhausting characterization with respect to optimality can be given. Precisely, for all qubit-qunit decomposable entanglement witnesses the three statements are equivalent: (i) W is optimal, (ii) $W = Q^{\Gamma}$ with Q being a positive operator supported on a completely entangled subspace (CES) and Γ denoting the partial transposition map[‡], (iii) the Hilbert space $\mathbb{C}^2 \otimes \mathbb{C}^n$ is spanned by product vectors obeying $\langle e, f | W | e, f \rangle = 0$. We achieve this goal by showing that product vectors orthogonal to any CES of $\mathbb{C}^2 \otimes \mathbb{C}^n$ after partial conjugation (PC)[§] span $\mathbb{C}^2 \otimes \mathbb{C}^n$. This means that (ii) implies (iii) and together with already proven facts that (iii) implies (i) and (i) implies (ii) [12], gives the above equivalence. The above fact also solves, at least in this particular case, the long-standing question whether (i) implies (iii).

[‡] Note that we do not specify the subsystem on which the transposition map is applied since our results are independent of this choice. However, for convenience, in all the proofs the transposition map is applied to the lower-dimensional subsystem.

[§] By the partial conjugation of a product vector $|e, f\rangle$ we mean the complex conjugation of either $|e\rangle$ or $|f\rangle$. Since our result do not depend on the choice of the subsystem subject to PC, we do not state explicitly on which subsystem it acts. Nevertheless, for convenience, in all the proofs, it is applied to the lower-dimensional subsystem.

Then we study DEWs acting on higher-dimensional Hilbert spaces and show that already in the simplest case of $3 \otimes 3$ the above equivalence appears to be false. Specifically, depending on the rank of Q , (ii) does not always imply (iii). This in turn implies that either not all witnesses admitting the form $W = Q^\Gamma$ are optimal ((ii) does not imply (i)), or not all OEWs have the property that product vectors satisfying $\langle e, f | W | e, f \rangle = 0$ span the corresponding Hilbert space ((i) does not imply (iii)).

It should be noticed that in the case of indecomposable EWs (IEW), examples of witnesses for which (i) does not imply (iii) are already known. A particular example of such a witness comes from the Choi map [23]. The latter is extremal in the convex set of positive maps [24] and therefore gives a optimal EW (see e.g. Ref. [22]). On the other hand, product vectors from $\mathbb{C}^3 \otimes \mathbb{C}^3$ at which the witness has zero mean value span a seven dimensional subspace in $\mathbb{C}^3 \otimes \mathbb{C}^3$ (see Refs. [18, 21]). Recently, using the theory of convex cones, the geometrical properties of such witnesses have been studied in Ref. [21].

The paper is organized as follows. In Sec. 2 we recall all the necessary notions and present, in a concise way, all we need about optimality of DEW. Then, in Sec. 3, we present our main results. We conclude in Sec. 4.

2. Preliminaries

For further benefits let us now recall some definitions and facts regarding decomposable entanglement witnesses. We give the definitions of separable states, entanglement witnesses, optimal and decomposable entanglement witnesses. Then, we shortly remind what is known regarding relations between optimality and decomposability of EWs.

In what follows we will be concerned with finite-dimensional product Hilbert spaces $\mathbb{C}^m \otimes \mathbb{C}^n$, henceforward denoted shortly by $\mathcal{H}_{m,n}$. By $\mathcal{D}_{m,n}$ and $\mathcal{D}_{m,n}^{\text{sep}}$ we denote, respectively, the set of all density matrices and separable density matrices acting on $\mathcal{H}_{m,n}$. In the case of equal local dimensions $m = n$, we use a single subscript m . Finally, $M_m(\mathbb{C})$ will denote the set of $m \times m$ matrices with complex entries.

Following Ref. [25], we call a density matrix ϱ acting on $\mathcal{H}_{m,n}$ *separable* if it can be written as

$$\varrho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1, \quad (1)$$

where $|a_i\rangle$ and $|b_i\rangle$ denote some pure states from \mathbb{C}^m and \mathbb{C}^n , respectively.

In 1996, basing on the Hahn-Banach separation theorem (cf. [26]), an important fact regarding the separability problem was proven [1]. Namely, a state ϱ acting on $\mathcal{H}_{m,n}$ is entangled if and only if there exists a Hermitian operator $W \in M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ such that $\text{Tr}(W\varrho) < 0$ and at the same time $\text{Tr}(W\sigma) \geq 0$ for all $\sigma \in \mathcal{D}_{m,n}^{\text{sep}}$. This fact gives rise to the following definition. Any Hermitian operator W acting on $\mathcal{H}_{m,n}$ is called *entanglement witness* if it has the properties: (i) its mean value $\langle W \rangle_\sigma$ in any $\sigma \in \mathcal{D}_{m,n}^{\text{sep}}$ is nonnegative, (ii) there exists an entangled state σ such that $\langle W \rangle_\sigma < 0$. Notice, that both the conditions can be rephrased as follows: (i) $\langle e, f | W | e, f \rangle \geq 0$ for any pair of vectors $|e\rangle \in \mathbb{C}^m$ and $|f\rangle \in \mathbb{C}^n$, (ii) W has at least one negative eigenvalue.

Now, via the Choi-Jamiołkowski isomorphism [27, 28], the theory of positive maps induces the following partition of EWs [29, 12]. An entanglement witness W is called

decomposable (DEW) if it can be written as $W = aP + (1 - a)Q^\Gamma$ with $P, Q \geq 0$ and $a \in [0, 1]$. EWs that do not admit this form are called *indecomposable*. Notice that the decomposable witnesses detect only states which have nonpositive partial transposition (NPT). For detection of entangled states with positive partial transposition we need to use indecomposable entanglement witnesses.

Let us now pass to the notion of optimality. To this aim we introduce

$$D_W = \{\varrho \in \mathcal{D}_{m,n} | \langle W \rangle_{\varrho} < 0\}, \quad (2)$$

that is, the set of all entangled states detected by W . Following Ref. [12], we say that given two EWs W_i ($i = 1, 2$), W_1 is *finer* than W_2 if $D_{W_2} \subseteq D_{W_1}$. Then, we say that W is *optimal* if there does not exist any other entanglement witness which is finer than W .

It was shown in Ref. [12] that W_1 is finer than W_2 if and only if there exists a positive number ϵ and a positive operator P such that W_1 can be expressed as $W_1 = (1 - \epsilon)W_2 + \epsilon P$. This immediately implies that W is optimal iff for any $\epsilon > 0$ and $P \geq 0$, the operator $\tilde{W} = (1 + \epsilon)W - \epsilon P$ is not an EW. The only candidates for positive operators that can be subtracted from W according to the above recipe must obey $PP_W = 0$, with

$$P_W = \{|e, f\rangle \in \mathcal{H}_{m,n} | \langle e, f | W | e, f \rangle = 0\}. \quad (3)$$

This implies a sufficient criterion for optimality of EWs. Namely, if the set of product vectors P_W spans the Hilbert space $\mathcal{H}_{m,n}$, the witness W is optimal. Eventually, application of the above facts to the general form of DEW allows us to conclude that if a decomposable EW is optimal, it has to be of the form

$$W = Q^\Gamma, \quad Q \geq 0, \quad (4)$$

where $\text{supp}(Q)$ does not contain any product vectors, or, in other words, $\text{supp}(Q)$ is a completely entangled subspace (CES) in $\mathcal{H}_{m,n}$.

3. Optimality and product vectors in subspaces orthogonal to completely entangled subspaces

From the preceding section we know that regarding optimality of decomposable EWs, two facts hold: (i) if a DEW W is optimal, then it has to have the form (4) and (ii) if P_W corresponding to W spans $\mathcal{H}_{m,n}$, then W is optimal. One could then ask if the opposite statements are also true. Or, in other words, if optimality of W is equivalent to the form (4), or to the fact that P_W spans $\mathcal{H}_{m,n}$.

First we show that in the case of the Hilbert space $\mathcal{H}_{2,n}$ the fact that DEW W can be written as in Eq. (4), implies that P_W spans $\mathcal{H}_{2,n}$. This immediately implies that both the above equivalences hold. On the other hand, we show that already in the $3 \otimes 3$ case there are witnesses admitting the form (4), but the P_W does not span \mathcal{H}_3 . Consequently, one of the above equivalences cannot hold. Either not all DEWs of the form (4) are optimal, or optimality does not imply that P_W spans \mathcal{H}_3 .

Before we start with our proofs, let us notice that since we deal only with witnesses that admit the form (4), the question about properties of P_W can be seen as the question about properties of product vectors orthogonal to completely entangled subspaces. This is a consequence of a simple property of the transposition map saying that its dual map is again the transposition map, which allows to conclude that $\langle e, f | Q^\Gamma | e, f \rangle = \langle e^*, f | Q | e^*, f \rangle$ any product vector $|e, f\rangle \in \mathcal{H}_{m,n}$. This together with

positivity of Q allows to conclude that $|e, f\rangle$ belongs to P_W iff $|e^*, f\rangle \in \ker(Q)$. Thus, in what follows we can ask a bit more general question, namely if partially conjugated product vectors orthogonal to a given CES, span the corresponding Hilbert space. For instance, we will show that for any CES V of $\mathbb{C}^2 \otimes \mathbb{C}^n$, the product vectors belonging to V^\perp span V^\perp , while their partial conjugations span V . Notice that completely entangled subspaces were recently investigated e.g. in [30, 31, 32, 33]. In particular, it was shown that the maximal dimension of CES in $\mathcal{H}_{m,n}$ is $(m-1)(n-1)$. This translates to the upper bound on the rank of Q in (4), i.e., $r(Q) \leq (m-1)(n-1)$.

We still need to introduce some more terminology. We say that a positive operator Q is supported on $\mathcal{H}_{m,n}$ if Q_A and Q_B have ranks m and n . Otherwise, if either Q_A or Q_B contains some vectors in its kernel, the operator Q acts effectively on a Hilbert space with smaller dimension. This can be translated to subspaces of $\mathcal{H}_{m,n}$. We can say that a given V is supported in $\mathcal{H}_{m,n}$ if the latter is the "smallest" Hilbert space of which V can be a subspace. In other words, projector onto this subspace is supported on $\mathcal{H}_{m,n}$.

Eventually, let V be a subspace of some Hilbert space \mathcal{H} . By V^\perp we will be denoting the subspace of \mathcal{H} of all vectors orthogonal to V (complement of V in \mathcal{H}). Also, the notation

$$\mathbb{C}^n \ni |f\rangle = \sum_i f_i |i\rangle \equiv (f_0, f_1, \dots, f_{n-1}) \quad (5)$$

will be frequently used.

3.1. Decomposable witnesses acting on $\mathcal{H}_{2,n}$

Let us first concentrate on the simplest case of $m = 2$. It follows from the previous discussion that the maximal dimension of a completely entangled subspace of $\mathbb{C}^2 \otimes \mathbb{C}^n$ is $n-1$. For pedagogical purposes let us start our considerations with this case. Then, we will pass to the cases of the remaining possible dimensions.

Lemma 1. *Let V be a $(n-1)$ -dimensional CES of $\mathbb{C}^2 \otimes \mathbb{C}^n$. Then there exists a nonsingular $n \times n$ matrix A such that the family of product vectors*

$$|e(\alpha), f(\alpha)\rangle \equiv (1, \alpha) \otimes A(1, \alpha, \dots, \alpha^{n-1}) \quad (\alpha \in \mathbb{C}) \quad (6)$$

belong to V^\perp . Moreover, the vectors $|e(\alpha), f(\alpha)\rangle$ ($\alpha \in \mathbb{C}$) span V^\perp , while $|e^(\alpha), f(\alpha)\rangle$ span $\mathbb{C}^2 \otimes \mathbb{C}^n$.*

Proof. Let $|\Psi_i\rangle$ ($i = 1, \dots, n-1$) denote the linearly independent vectors spanning V . All of them have to be entangled as otherwise there would exist a product vector in V . This means that they can be expressed as

$$|\Psi_i\rangle = |0\rangle|\psi_0^{(i)}\rangle + |1\rangle|\psi_1^{(i)}\rangle \quad (i = 1, \dots, k), \quad (7)$$

with nonzero vectors $|\psi_j^{(i)}\rangle$ ($i = 1, \dots, k; j = 0, 1$) such that the vectors in each pair $|\psi_j^{(i)}\rangle$ ($j = 0, 1$) are linearly independent. Also, it is easy to see that the vectors in both the sets, $\{|\psi_0^{(i)}\rangle\}_{i=1}^{n-1}$ and $\{|\psi_1^{(i)}\rangle\}_{i=1}^{n-1}$, are linearly independent. Otherwise in both cases it is possible to find a product vector in V .

Let us now look for the product vectors $|e, f\rangle$ orthogonal to V , where we take $|e\rangle = (1, \alpha) \in \mathbb{C}^2$ with $\alpha \in \mathbb{C}$ and arbitrary $|f\rangle = (f_0, \dots, f_{n-1}) \in \mathbb{C}^n$. The

By Q_A and Q_B we denote $\text{Tr}_B(Q)$ and $\text{Tr}_A(Q)$, respectively. Notice that both are positive.

orthogonality conditions to $|\Psi_i\rangle$ ($i = 1, \dots, n-1$) give us the set of $n-1$ linear homogenous equations

$$\langle \Psi_i | e, f \rangle = 0 \quad (i = 1, \dots, n-1). \quad (8)$$

for n variables f_i . In order to solve it, we can fix one of the variables, say $f_0 = 1$, getting a system of $n-1$ inhomogenous equations for $n-1$ variables. It can easily be solved and the solution is given by

$$f_i(\alpha) = \frac{R_i(\alpha)}{R(\alpha)} \quad (i = 1, \dots, n-1), \quad (9)$$

where R_i and R are polynomials in α of degree at most $n-1$. Moreover, since the vectors $|\psi_1^{(i)}\rangle$ ($i = 1, \dots, n$) are linearly independent, the degree of the polynomial R is exactly $n-1$. Consequently, the product vectors in V^\perp we look for, take the generic form

$$|e(\alpha), f(\alpha)\rangle = (1, \alpha) \otimes (R(\alpha), R_1(\alpha), \dots, R_{n-1}(\alpha)) \quad (\alpha \in \mathbb{C}). \quad (10)$$

Note that we have multiplied above everything by $R(\alpha)$, so that the expression (10) is valid also for $R(\alpha) = 0$, while the expression (9) only when $R(\alpha) \neq 0$. Nevertheless, by continuity or by local change of the basis one shows that the vectors (10) for α being the roots of R are also orthogonal to $|\Psi_i\rangle$.

For further purposes, let us denote by V_{sep}^\perp the subspace of V^\perp spanned by all the vectors (10).

The assumption that V does not contain any product vector implies that all the polynomials R, R_i are linearly independent. In order to see it explicitly, let us assume that only $k < n$ of them are linearly independent. Then, there has to exist $n-k$ vectors $|\xi_i\rangle$ ($i = 1, \dots, n-k$) that are orthogonal to the subspace of \mathbb{C}^n spanned by $|f(\alpha)\rangle$ ($\alpha \in \mathbb{C}$). Moreover, for any $|h\rangle \in \mathbb{C}^2$, vectors $|h\rangle|\xi_i\rangle$ ($i = 1, \dots, n-k$) are orthogonal to V_{sep}^\perp . In what follows we show that among the latter there exists at least one product vector which is orthogonal to V^\perp and thus has to be in V leading to the contradiction with the assumption that V is a CES.

For this purpose, let us notice that vectors (10) span $(k+1)$ -dimensional subspace in V^\perp . As a result, there exists a set of $n-k$ vectors $|\omega_i\rangle \in V^\perp$ ($i = 1, \dots, n-k$), which are orthogonal to all $|e(\alpha), f(\alpha)\rangle$. Now, we take the following product vector

$$|\eta\rangle = (|0\rangle + \gamma|1\rangle) \otimes \sum_{i=1}^{n-k} b_i |\xi_i\rangle \quad (11)$$

with $\gamma \in \mathbb{C}$ and $b_i \in \mathbb{C}$ being some parameters to be determined. Obviously, $|\eta\rangle$ is already orthogonal to V_{sep}^\perp . Orthogonality conditions $\langle \omega_i | \eta \rangle = 0$ ($i = 1, \dots, n-k$) give us the system of $n-k$ homogenous equations for $n-k$ variables b_i of the form $(M_1 + \gamma M_2)|b\rangle = 0$ with $|b\rangle = (b_1, \dots, b_{n-k})$ and M_i being some matrices. It has a nontrivial solution only if $\det(M_1 + \gamma M_2)$ vanishes. The latter is a polynomial in γ of at most $(n-k)$ th degree and obviously the corresponding equation is soluble in the complex field. Consequently, we have product vectors belonging to V , which is in a contradiction with the assumption that V is a CES. Thus, R, R_i ($i = 1, \dots, n-1$) are linearly independent. This in turn means that there exists a nonsingular transformation $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $|f(\alpha)\rangle = A(1, \alpha, \dots, \alpha^{n-1})$ for any $\alpha \in \mathbb{C}$.

On the other hand, it is easy to see that vectors

$$(1, \alpha) \otimes (1, \alpha, \dots, \alpha^{n-1}) \quad (\alpha \in \mathbb{C}) \quad (12)$$

span $(n+1)$ -dimensional subspace of $\mathbb{C}^2 \otimes \mathbb{C}^n$, while their PCs, that is,

$$(1, \alpha^*) \otimes (1, \alpha, \dots, \alpha^{n-1}) \quad (\alpha \in \mathbb{C}), \quad (13)$$

span the whole $\mathbb{C}^2 \otimes \mathbb{C}^n$. In the first case this is because among $2n$ monomials in α appearing in Eq. (12), $n+1$ are linearly independent. In the second case, α^* is linearly independent of any polynomial in α and thus we have $2n$ linearly independent polynomials in (13). Therefore, since A is of full rank, vectors $|e(\alpha)\rangle \otimes A|f(\alpha)\rangle$ ($\alpha \in \mathbb{C}$) span V , while $|e^*(\alpha)\rangle \otimes A|f(\alpha)\rangle$ the whole $\mathcal{H}_{2,n}$. This finishes the proof. ■

Let us now pass to the remaining cases with respect to the dimension of V .

Lemma 2. *Let V be a $k < n-1$ -dimensional CES of $\mathbb{C}^2 \otimes \mathbb{C}^n$. Then there exists a nonsingular transformation A , such that the vectors*

$$(1, \alpha) \otimes A(R(\alpha, \beta), \beta_1 R(\alpha, \beta), \dots, \beta_{n-k-1} R(\alpha, \beta), R_1(\alpha, \beta), \dots, R_k(\alpha, \beta)) \quad (\alpha, \beta_1, \dots, \beta_{n-k-1} \in \mathbb{C}) \quad (14)$$

span V^\perp , while their PCs span $\mathbb{C}^2 \otimes \mathbb{C}^n$. Here $\beta \equiv (\beta_1, \dots, \beta_{n-k-1})$, $R(\alpha, \beta)$ and $R_i(\alpha, \beta)$ are polynomials of at most k th degree in α and first degree in β_i ($i = 1, \dots, n-k-1$).

Proof. We can follow the same reasoning as in the proof of Lemma 1. Now, we have k entangled vectors $|\Psi_i\rangle$ spanning V which can be written as in Eq. (7). For the same reason as before both sets $\{|\psi_0^{(i)}\rangle\}_{i=0}^{k-1}$ and $\{|\psi_1^{(i)}\rangle\}_{i=0}^{k-1}$ are linearly independent. Therefore we can always find a nonsingular transformation $\tilde{A} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\tilde{A}|\psi_1^{(i)}\rangle = |n-k-1+i\rangle$ ($i = 1, \dots, k$).

Let us now consider the locally transformed subspace $\tilde{V} = (\mathbb{1}_2 \otimes \tilde{A})V(\mathbb{1}_2 \otimes \tilde{A}^\dagger)$, which is also a CES, and look for the separable vectors belonging to \tilde{V}^\perp and taking the following form

$$(1, \alpha) \otimes (1, \beta_1, \dots, \beta_{n-k-1}, f_1, \dots, f_k), \quad (15)$$

where $\beta_i \in \mathbb{C}$ are free parameters and f_i ($i = 1, \dots, k$) are to be determined. Orthogonality conditions to k vectors spanning \tilde{V} , i.e., $|\tilde{\Psi}_i\rangle = \mathbb{1}_2 \otimes \tilde{A}|\Psi_i\rangle$, lead us to the following inhomogenous linear equations

$$\sum_{j=1}^k f_j \left(\langle \tilde{\psi}_0^{(i)} | n-k-1+j \rangle + \alpha \delta_{ij} \right) = x_i(\alpha, \beta) \quad (i = 1, \dots, k), \quad (16)$$

where $x_i(\alpha, \beta)$ are polynomials of the first degree in α and all β s.

Following the same reasoning as in the proof of lemma 1, one obtains the product vectors orthogonal to $|\tilde{\Psi}_i\rangle$ in the form

$$|e(\alpha), f(\alpha, \beta)\rangle = (1, \alpha) \otimes (R, \beta_1 R, \dots, \beta_{n-k-1} R, R_1, \dots, R_k) \quad (\alpha, \beta_i \in \mathbb{C}), \quad (17)$$

where R_i and R are polynomials of degree at most k in α and one in β s (for brevity we omitted arguments of R and R_i in (17)). Moreover, due to the already mentioned fact that the vectors $|\psi_1^{(i)}\rangle$ ($i = 0, \dots, k-1$) are linearly independent, the highest power of α in R is exactly k .

Let us now show that the polynomials R , $\beta_i R$ ($i = 1, \dots, n-k-1$), and R_i ($i = 1, \dots, k$) are linearly independent. For this purpose, let us assume that only $m < n$ of them are linearly independent. It is clear that $m \geq n-k$ as the monomials 1 and β_i ($i = 1, \dots, n-k-1$) are by the very definition linearly independent and

therefore we can denote $m = n - k + l$ with $l = 1, \dots, k$. Consequently, there exist $k - l$ vectors $|\tilde{\xi}_i\rangle \in \mathbb{C}^n$ orthogonal to the subspace spanned by $|f(\alpha, \beta)\rangle$ ($\alpha, \beta_i \in \mathbb{C}$).

On the other hand, since R is of k th degree in α and the above m polynomials are of degree at most k in α , they, together with $n - k$ polynomials $\alpha R(\alpha, \beta)$ and $\alpha \beta_i R(\alpha, \beta)$ ($i = 1, \dots, n - k - 1$), constitute the set of $2(n - k) + l$ linearly independent polynomials. This implies that the vectors (17) span at least $2(n - k) + l$ -dimensional subspace in \tilde{V}^\perp . In the worst case scenario, i.e., when this dimension is exactly $2(n - k) + l$ we have $k - l$ linearly independent $|\tilde{\omega}_i\rangle \in \tilde{V}^\perp$ which are orthogonal to all vectors (17). Then, following the same reasoning as in the proof of lemma 1, we can show that there are product vectors in \tilde{V} , which contradicts the fact that \tilde{V} is CES.

In conclusion, all the polynomials $R, \beta_i R$ ($i = 1, \dots, n - k - 1$) and R_i ($i = 1, \dots, k$) are linearly independent. As a result, these n polynomials together with $n - k$ polynomials αR and $\alpha \beta_i R$ ($i = 1, \dots, n - k - 1$) constitute the set of $2n - k$ linearly independent polynomials and therefore the continuous set of product vectors $|e(\alpha), f(\alpha, \beta_i)\rangle$ in Eq. (17) span \tilde{V} . Also, for the same reason as before, the partially conjugated vectors

$$(1, \alpha^*) \otimes (R, \beta_1 R, \dots, \beta_{n-k-1} R, R_1, \dots, R_k) \quad (18)$$

span $\mathcal{H}_{2,n}$.

Eventually, putting $A = (\tilde{A}^{-1})^\dagger$, we see that the vectors (14) span V^\perp , while their PCs span $\mathcal{H}_{2,n}$. This completes the proof. ■

The above lemmas together with the previously known results allow us to prove the following theorem.

Theorem 1. *Let W be a decomposable witness acting on $\mathcal{H}_{2,n}$. The the following statements are equivalent:*

- (i) $W = Q^\Gamma$, where $Q \geq 0$ and $\text{supp}(Q)$ is a CES in $\mathcal{H}_{2,n}$,
- (ii) P_W spans $\mathbb{C}^2 \otimes \mathbb{C}^n$,
- (iii) W is optimal.

Proof. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ were proven in Ref. [12]. The implication $(i) \Rightarrow (ii)$ follows from the above lemmas. ■

Let us illustrate the above discussion with a simple example. Let us consider a witness $W = Q^\Gamma$ with Q supported on a $(n - 1)$ -dimensional subspace V of $\mathbb{C}^2 \otimes \mathbb{C}^n$ spanned by the following vectors

$$|\Psi_i\rangle = (1/\sqrt{2})(|0, i\rangle - |1, i - 1\rangle) \quad (i = 1, \dots, n - 1) \quad (19)$$

The subspace V does not contain any product vector because, as one can directly check, there does not exist product vector orthogonal to $V^\perp = \text{span}\{|00\rangle, |1, n - 1\rangle, (1/\sqrt{2})(|0, i\rangle + |1, i - 1\rangle) \ (i = 1, \dots, n - 1)\}$. Then the separable vectors spanning V^\perp are given by (12) and, as already mentioned, they span $\mathbb{C}^2 \otimes \mathbb{C}^n$.

3.2. Decomposable witnesses acting on \mathcal{H}_3

Here we show that the simple characterization we proved in theorem 1 for $2 \otimes n$ decomposable witnesses does not hold for some of witnesses acting already on $\mathbb{C}^3 \otimes \mathbb{C}^3$. Precisely, we will see that for witnesses (4) with $r(Q) = 1, 2$, the analog of the above

theorem also holds, while there are witnesses with $r(Q) = 3, 4$ such that the separable vectors from the corresponding P_W s do not span \mathcal{H}_3 .

Let us start from the case of $r(Q) = 1$. Here we have a bit more general fact (see also Ref. [22] for a proof of optimality via extremality). Then, we will consider the case of $r(Q) = 2$.

Lemma 3. *Let $W = |\psi\rangle\langle\psi|^\Gamma$, where $|\psi\rangle$ is an entangled pure state from \mathcal{H}_m . Then, the statements (i), (ii), and (iii) from theorem 1 (accordingly reformulated) are equivalent.*

Proof. As previously, implications (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) follow from Ref. [12]. Below we prove that (i) implies (ii).

The Schmidt decomposition of $|\psi\rangle$ reads

$$|\psi\rangle = \sum_{i=0}^{s-1} \sqrt{\mu_i} |ii\rangle, \quad (20)$$

where $\mu_i \geq 0$ and $s \leq m$ denotes the Schmidt rank of $|\psi\rangle$. Without any loss of generality we can assume that $s = m$. Then, by a local full rank transformation we can bring $|\psi\rangle$ to the maximally entangled state $|\psi_m^+\rangle$. The product vectors orthogonal to the latter are of the form $|e\rangle|f\rangle$, where $|e\rangle$ is arbitrary vector from \mathbb{C}^m and $|f\rangle \in \mathbb{C}^m$ is any vector orthogonal to $|e\rangle$. Then, this class of vectors after PC span $\mathbb{C}^m \otimes \mathbb{C}^m$ (cf. Ref. [38]). ■

Note, that with a bit more effort the above lemma can be generalized to any witness $W = |\psi\rangle\langle\psi|^\Gamma$ acting on $\mathbb{C}^m \otimes \mathbb{C}^n$.

Lemma 4. *Let V be a CES of \mathcal{H}_3 with $\dim V = 2$. Then the product vectors from V^\perp , when partially conjugated, span \mathcal{H}_3 .*

Proof. Let $|\Psi_i\rangle$ ($i = 1, 2$) be two linearly independent vectors spanning V . Clearly, we can assume that at least one of these vectors, say $|\Psi_1\rangle$, is of Schmidt rank two. By a local unitary operations it can be brought to $|\tilde{\Psi}_1\rangle = |00\rangle + |11\rangle$.

Let us now look for the product vectors orthogonal to V of the form $(1, \alpha, \beta) \otimes (f_0, f_1, f_2)$ ($\alpha, \beta \in \mathbb{C}$). From the orthogonality conditions to the transformed vectors $|\tilde{\Psi}_i\rangle$ ($i = 1, 2$) one infers that they take the form

$$(1, \alpha, \beta) \otimes (-\alpha R(\alpha, \beta), R(\alpha, \beta), R_1(\alpha, \beta)), \quad (21)$$

with R and R_1 being polynomials in α and β . Let us now show that the three polynomials R , αR , and R_1 are linearly independent (the first two already are). To this end we can follow the approach already used in the previous lemmas. Assume that R_1 is linearly dependent on R and αR . Then, there exist a vector $|\xi\rangle \in \mathbb{C}^3$ orthogonal to every $(-\alpha R(\alpha, \beta), R(\alpha, \beta), R_1(\alpha, \beta))$ ($\alpha, \beta \in \mathbb{C}$) and consequently any vector $|h\rangle|\xi\rangle$ with arbitrary $|h\rangle \in \mathbb{C}^3$ is orthogonal to the vectors (21). On the other hand, one immediately sees that the latter span five-dimensional subspace in V^\perp . This means that since $\dim V^\perp = 7$, there exist two vectors $|\omega_i\rangle \in V^\perp$ ($i = 1, 2$) orthogonal to all vectors in (21). It is then clear that among the two-parameter class $|h\rangle|\xi\rangle$ there exist at least one vector orthogonal to both $|\omega_i\rangle$ ($i = 1, 2$), implying the existence of a product vector in V . This is, however, in a contradiction with the assumption that V is a CES.

Since then R , αR , and αR are linearly independent, the partially conjugated vectors

$$(1, \alpha^*, \beta^*) \otimes (-\alpha R(\alpha, \beta), R(\alpha, \beta), R_1(\alpha, \beta)) \quad (\alpha, \beta \in \mathbb{C}) \quad (22)$$

certainly span \mathcal{H}_3 . ■

Basing on the above lemma 3 and lemma 4, we can now formulate the analog of theorem 1 for some of DEWs acting on \mathcal{H}_3 .

Theorem 2. *Let W be a decomposable witness acting on \mathcal{H}_3 . Then the following conditions are equivalent:*

- (i) $W = Q^\Gamma$ with $Q \geq 0$ such that $r(Q) = 1, 2$ and $\text{supp}(Q)$ being a CES,
- (ii) P_W spans \mathcal{H}_3 ,
- (iii) W is optimal.

Proof. The implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ were proven in Ref. [12]. The implication $(i) \Rightarrow (ii)$ follows from lemma 3 and 4. ■

Still the cases of $\dim V = 3, 4$ remain untouched. As we will see shortly, it is possible to provide examples of three and four-dimensional CESs supported in \mathcal{H}_3 such that their complements, V^\perp s, contain product vectors, which, when partially conjugated, do not span \mathcal{H}_3 . While, due to the fact that five-dimensional subspaces of \mathcal{H}_3 have generically six product vectors (cf. [33, 35]), the existence of such three-dimensional CESs for which the product vectors from their complements do not, when partially transposed, span \mathcal{H}_3 is surprising and interesting. This implies that there are DEWs (4) with $r(Q) = 3$ such that P_W s, even if containing continuous classes of product vectors, do not span \mathcal{H}_3 . Among such EWs one may look for the analogs of the aforementioned Choi-like witnesses (optimal witnesses whose P_W s do not span the corresponding Hilbert space) already known to exist among the indecomposable EWs [23]. Still, however, we cannot prove their optimality. On the other hand, it is possible to provide examples of witnesses (4) (thus also CESs) with $r(Q) = 3, 4$ such that their P_W s do span \mathcal{H}_3 .

In the first, three-dimensional case let us consider the subspace V_1 spanned by the following (unnormalized) vectors

$$|01\rangle + |10\rangle, \quad |02\rangle + |20\rangle, \quad |1\rangle(a|1\rangle + b|2\rangle) + |2\rangle(a_2|1\rangle + b_2|2\rangle). \quad (23)$$

For complex a_i, b_i ($i = 1, 2$) satisfying $ab_2 \neq a_2b$, V_1 does not contain any product vector. Then, under the conditions that $(a_2 + b)^2 = 4ab_2$ and $b_2 \neq 0$, one shows that all the product vectors in V_1^\perp are of the form

$$(1, \alpha, \lambda\alpha) \otimes (1, -\alpha, -\lambda\alpha) \quad (24)$$

and

$$(0, 1, \alpha) \otimes (0, b + b_2\alpha, -a - a_2\alpha), \quad (25)$$

where $\lambda = -(b + a_2)/2b_2$. Direct check allows to conclude that both classes after the partial conjugation span only a seven-dimensional subspace in \mathcal{H}_3 . Finally, since there do not exist PPT entangled states acting on \mathcal{H}_3 of rank three, any positive Q with $r(Q) = 3$ and supported on this subspace has to be NPT, thus giving rise to a proper witness.

In the four-dimensional case the problem of existence of EWs (4) for which P_W s do not span the corresponding Hilbert space is very much related to the results of [33, 35]. In particular, five-dimensional subspaces in \mathcal{H}_3 contain generically six product vectors (five of them are linearly independent), and obviously cannot span, when partially conjugated, \mathcal{H}_3 . In order to provide a particular example of an EW (4) with $r(Q) = 4$, one may consider a CES orthogonal to some unextendible product basis (UPB)¶ [36] (see also Ref. [34]). To this end, let us take one of the five-elements UPBs from \mathcal{H}_3 given in [36], called PYRAMID:

$$|\psi_i\rangle = |\phi_i\rangle|\phi_{2i \bmod 5}\rangle \quad (i = 0, \dots, 4) \quad (26)$$

with

$$|\phi_i\rangle = N \left[\cos\left(\frac{2\pi i}{5}\right) |0\rangle + \sin\left(\frac{2\pi i}{5}\right) |1\rangle + h_+ |2\rangle \right], \quad (27)$$

where $h_{\pm} = (1/2)\sqrt{\sqrt{5} \pm 1}$ and $N = 2/\sqrt{5 + \sqrt{5}}$. The subspace orthogonal to these vectors is spanned by orthogonal vectors of Schmidt rank two given by

$$\begin{aligned} \eta|10\rangle + (\eta|0\rangle + 2h_-^2|2\rangle)|1\rangle, & \quad -\eta|10\rangle + |1\rangle(\eta|0\rangle + 2h_-^2|2\rangle), \\ (-h_-|0\rangle + 4h_-^2|2\rangle)|0\rangle - h_-|11\rangle, & \quad |0\rangle(h_-|0\rangle + 4h_-^2|2\rangle) + h_-|11\rangle, \end{aligned} \quad (28)$$

where $\eta = 1/2h_+$.

Taking convex combination of projectors onto this vectors, denoted \mathcal{P}_i ($i = 1, 2, 3, 4$), with equal weights we obviously get PPT entangled state. However, by appropriate changing these weights we get a positive operator Q which is NPT. For instance we can consider the following one-parameter family of Q s

$$Q(r) = r(\mathcal{P}_1 + \mathcal{P}_2) + (1/2)(1 - 2r)(\mathcal{P}_3 + \mathcal{P}_4) \quad (0 \leq r \leq 1/2). \quad (29)$$

It is easy to check that $Q(r)$ is NPT except for $r = 1/4$.

In spite of the above examples it is still possible to provide three and four-dimensional CESs such that the product vectors in their complements do span, after PC, \mathcal{H}_3 . Note that generically for the three-dimensional CESs of \mathcal{H}_3 this is the case. Let us then consider the following subspace:

$$V_2 = \text{span}\{|01\rangle - |10\rangle, |02\rangle - |20\rangle, |12\rangle - |21\rangle, |02\rangle + |20\rangle - |11\rangle\}. \quad (30)$$

Note that V_2 contains the antisymmetric subspace of \mathcal{H}_3 and the fourth vector spanning it (which is of Schmidt rank three) belongs to the symmetric subspace of \mathcal{H}_3 . It is clear that V_2 is supported in \mathcal{H}_3 and it does not contain any product vectors. In order to see it explicitly, assume that some $|e, f\rangle$ can be written as a linear combination of all these vectors. Application of the swap operator to $|e, f\rangle$ gives $|f, e\rangle$. On the other hand, it changes the sign before first three vectors spanning V_2 and therefore one sees that $|02\rangle + |20\rangle - |11\rangle$ is proportional to $|e, f\rangle + |f, e\rangle$ which contradicts the fact that it has Schmidt rank three.

It is now easy to see that the product vectors

$$(1, \alpha, \alpha^2/2) \otimes (1, \alpha, \alpha^2/2) \quad (\alpha \in \mathbb{C}) \quad (31)$$

are orthogonal to V_2 and their PC span \mathcal{H}_3 .

¶ Following [36] we say that a set of product vectors from some product Hilbert space \mathcal{H} is unextendible product basis if the vectors are orthogonal and there does not exist any other product vector in \mathcal{H} orthogonal to all of them. Skipping the orthogonality condition we get nonorthogonal UPB.

4. Conclusion

Let us shortly summarize the obtained results and sketch lines of further possible research.

Entanglement witnesses give one of the most relevant tools in the theory of entanglement. Their characterization is therefore of a great interest. In this note we have focused on the simpler case of decomposable entanglement witnesses and investigated couple of issues related to the notion of optimality. In the $2 \otimes n$ case, more profound characterization can be given to DEWs. Together with Ref. [12], our results show that a given DEW W is optimal iff the corresponding P_W spans $\mathcal{H}_{2,n}$. Then, the latter holds iff $W = Q^\Gamma$ with positive Q supported on some CES. Interestingly, such transparent characterization does not hold already in the case of DEWs acting on \mathcal{H}_3 . Precisely, although for all such DEWs with $r(Q) = 1, 2$ the above equivalences also hold, there exist DEWs with $r(Q) = 3, 4$ such that the product vectors from the corresponding PWs do not span \mathcal{H}_3 . This in general means that either not all witnesses taking the form (4) with Q supported on a CES are optimal, or that optimality of a DEWs W does not necessarily mean that its P_W spans the corresponding Hilbert space.

Obviously the obtained results do not complete the characterization of DEWs, even in the two-qutrit case. In particular, the complete analysis of the cases when $r(Q) = 3, 4$ is missing. Even if for $r(Q) = 3$, generically PWs of DEWs (4) span \mathcal{H}_3 , it is possible to find examples of DEWs, as the one provided above, for which this is not the case. One task would be to characterize such witnesses and check if some of them are optimal. This would prove that also in the case of DEWs optimality does not imply that P_W spans the Hilbert space on which W acts. Let us remind that the existence of indecomposable EWs having this property is already known [23, 18, 21].

On the other hand, in the case of $r(Q) = 4$ it follows from e.g. [33, 35] that almost all four-dimensional subspaces of \mathcal{H}_3 have only six product vectors in their complement meaning that generic P_W s of DEW (4) with $r(Q) = 4$ do not span \mathcal{H}_3 . Nevertheless there exist CESs, as for instance the one presented above, such that product vectors in their complement span \mathcal{H}_3 . Again, it seems interesting to characterize these CESs.

Then, one could ask the same questions in the case of higher-dimensional Hilbert spaces $\mathcal{H}_{m,n}$ and, finally, similar analysis is missing in the case of indecomposable entanglement witnesses.

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